OPTIMIZATION OF THE PROCESS OF GROWTH OF ELASTIC BODIES

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The optimal rate of flow of matter to a growing elastic body, ensuring minimum stresses or displacements at the moment the growth process ends, is found for an external load varying arbitrarily in time. The problem is solved in the quasistatic formulation for small deformations.

1. Formulation of the Problem of Optimization of the Process of Growth of a Column. We shall study the process of continuous growth of a column of linearly elastic material. Prior to deformation the column consists of a cylinder with length ℓ and a circular transverse cross section with radius a_0. The bottom end of the column is fastened rigidly, while the upper end is free. At the time t = 0 material starts to accrete on the lateral surface of the column, and equipment is placed on the top end. Because of the inflow of matter from outside the radius of the cylinder varies according to the law $a = a(t)$. The radius of the growing cylinder equals a_1 at the moment growth stops t = T. The function $a(t)$ does not decrease monotonically. We shall denote by $V(t) = 2\pi l a(t) a^{\cdot}(t)$ the rate of growth of the cylinder ($a^{\cdot} = da/dt$). The function V(t) is bounded: $0 \leq V_1 \leq V(t) \leq V_2 < \infty$ (V_1 and V_2 are the minimum and maximum rates of inflow of matter). Growth occurs freely.

The effect of the equipment on the column reduces to a compressive force P = P(t), applied to the end of the cylinder and equal to the weight of the equipment. We assume that $P(0) = 0$, $P(T) = P_0 \geq 0$. The quantity P(t) is not a monotonic function of time, since additional instruments and devices, necessary for setting and adjusting the equipment, could be placed during the growth process on the top of the column and then subsequently removed. The maximum achievable rate of lifting or removal of the equipment is fixed: $|P^{\cdot}(t)| \leq U_1$.

The external load creates a longitudinal strain of the column. We introduce the cylindrical coordinate system (r, ϑ, z), whose z axis is aligned with the longitudinal axis of the column. In a uniaxial stressed state the strain ε is related with the stress σ by the equality [1]

$$\sigma(t, r) = E[\varepsilon(t) - \varepsilon(\tau^*(r))], \tag{1.1}$$

where E is the constant modulus of elasticity and τ^* is the moment at which the column material nucleates. For the starting cylinder ($0 \leq r \leq a_0$) we assume that $\tau^* = 0$, and in the accreted region ($0 \leqslant r \leqslant a_1$) the function $t = \tau^*(r)$ is the inverse of $r = a(t)$.

The expression (1.1) satisfies the equations of equilibrium and the boundary conditions on the lateral surface of the cylinder. We write down the boundary conditions at the ends of the column: $2\pi \int\limits_0^{a(t)} \sigma(t, r) r dr = -P(t)$. We substitute (1.1) into this equality. We represent the integral over r as a sum of two integrals from 0 to a_0 and from a_0 to $a(t)$. We calculate the first integral explicitly, and in the second integral we change the variable of integration to $r = a(s)$. The result is

$$a_0^2 \varepsilon(t) + 2\int\limits_0^t [\varepsilon(t) - \varepsilon(s)]\, a(s)\, a^{\cdot}(s)\, ds = -P(t)(\pi E)^{-1}. \tag{1.2}$$

We differentiate the relation (1.2) with respect to the time t. Taking into account the relation between the radius of the growing cylinder $a(t)$ and the rate of accretion V(t), we write the equality obtained in the form

$$\varepsilon^{\cdot} = -u(t)\left[1 + \int\limits_0^t v(s)\, ds\right]^{-1}, \quad \varepsilon(0) = 0. \tag{1.3}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 156-160, November-December, 1988. Original article submitted September 25, 1987.

0021-8944/88/2906-0911$12.50

Here $u(t) = P^{\bullet}(t)(\pi E a_0^2)^{-1}$; $v(t) = V(t)(\pi \ell a_0^2)^{-1}$. We find the initial condition for Eq. (1.3), settling t = 0 in the relation (1.2).

The following constraints are imposed on the functions u(t) and v(t):

$$\int_0^T u(t)\,dt = p_0, \quad \int_0^t u(s)\,ds \geqslant 0, \quad 0 \leqslant t \leqslant T; \tag{1.4}$$

$$\int_0^T v(t)\,dt = \lambda, \tag{1.5}$$

where $p_0 = P_0(\pi E a_0^2)^{-1}$; $\lambda = (a_1 a_0^{-1})^2 - 1$.

The problem of optimizing the process of growth of the column consists of determining a piecewise-continuous rate of accretion v(t) satisfying the inequality

$$v_1 \leqslant v(t) \leqslant v_2 \left(v_1 = V_1(\pi l a_0^2)^{-1}, \quad v_2 = V_2(\pi l a_0^2)^{-1}\right) \tag{1.6}$$

and giving a minimum value for the longitudinal strain at the moment accretion stops $|\varepsilon(T)|$. Since $\varepsilon(T)$ depends on the loading history P(t) we shall interpret this problem as a problem of determining

$$\inf_{v(t)} \sup_{u(t)} |\varepsilon(T)|. \tag{1.7}$$

The supremum in this expression is calculated over all piecewise-continuous functions u(t) for which the estimate

$$|u(t)| \leqslant u_1 \left(u_1 = U_1(\pi E a_0^2)^{-1}\right). \tag{1.8}$$

is valid.

In what follows we shall assume that the exact top and bottom boundaries in the expression (1.7) are reached for unique (to within the values on sets of measure zero) functions $u_0(t)$ and $v_0(t)$. In addition we assume that there exist a nontrivial selection of the optimal rate of accretion of the column and a nontrivial loading:

$$V_1 T < \pi(a_1^2 - a_0^2)\, l < V_2 T, \quad P_0 < U_1 T. \tag{1.9}$$

The relations (1.9) prevent the functions u(t) and v(t) from assuming only the maximum possible or only the minimum possible value for almost all $t \in [0, T]$. It is assumed that at any time $t \in [0, T]$ the compressive load P(t) does not exceed the Eulerian critical force and the column cannot become unstable.

2. Determination of the Optimal Loading Regime. We fix the admissible rate of accretion v(t) and study first the problem of determining the function $u_0(t)$, which gives the functional $\Phi = |\varepsilon(T)|$ a minimum value on the set of trajectories of Eq. (1.3) under the restrictions (1.4). It is easy to verify that for any admissible function u(t) the quantity $\varepsilon(T)$ is negative. We replace the problem of determining the maximum of the functional Φ by the problem of maximizing the functional $\Phi_1 = -\varepsilon(T) + \psi_1\left(\int_0^T u(t)\,dt - p_0\right)$ on the set of trajectories of Eq. (1.3). Here ψ_1 is a Lagrange multiplier, taking into account the first condition (1.4). If the function $u_0(t)$, giving an extremum to the functional Φ_1, guarantees that the second condition (1.4) holds, then this function also gives an extremum to the functional Φ. We shall calculate the increment to the function Φ_1:

$$\Delta\Phi_1 = \int_0^T F_1(t)\,\Delta u(t)\,dt \left(F_1(t) = \psi_1 + \left[1 + \int_0^t v(s)\,ds\right]^{-1}\right). \tag{2.1}$$

According to the necessary condition for optimality [2], for the function $u_0(t)$ the expression in the integrand in (2.1) must be nonpositive for almost all $t \in [0, T]$ and any admissible increments to the function $u_0(t)$. This condition holds, if the optimal function $u_0(t)$ assumes the value $-u_1$ for $F_1(t) < 0$ and u_1 for $F_1(t) \geq 0$. It follows from here and (1.9) that the function $F_1(t)$ cannot assume only negative or only nonnegative values on the segment [0, T]. Since the function $F_1(t)$ does not increase monotonically, there exists a $t_0 \in (0, T)$ such that

$$u_0(t) = u_1,\ 0 \leqslant t < t_0;\ u_0(t) = -u_1,\ t_0 \leqslant t \leqslant T. \tag{2.2}$$

The parameter t_0 is determined from (1.4) and (2.2) in the form

$$t_0 = (1/2)\left(T + p_0 u_1^{-1}\right). \tag{2.3}$$

According to (2.2) the second condition of (1.4) holds. The function $u_0(t)$ gives a maximum of the functional Φ and does not depend on the rate of accretion of material $v(t)$.

3. Determination of the Optimal Accretion Regime. In accordance with (2.2) the problem of optimizing the rate of accretion of the column consists of determining the optimum control by Eq. (1.3) with $u = u_0(t)$, which gives a minimum value to the functional Φ under the restriction (1.5). Introducing the Lagrange multipler ψ_2, which takes into account the isoperimetric condition (1.5), we replace this problem by the problem of minimizing the functional $\Phi_2 = -\varepsilon(T) + \psi_2\left(\int_0^T v(t)\,dt - \lambda\right)$ on the set of trajectories of Eq. (1.3). We shall calculate the increment to the functional Φ_2:

$$\Delta\Phi_2 = \int_0^T F_2(t)\,\Delta v(t)\,dt; \tag{3.1}$$

$$F_2(t) = \psi_2 - \int_t^T u_0(\tau)\left[1 + \int_0^\tau v(s)\,ds\right]^{-2} d\tau. \tag{3.2}$$

According to the necessary condition of optimality [2], for the function $v_0(t)$ the expression in the integrand in (3.1) is nonnegative for almost all $t \in [0, T]$ and for any admissible increments to the function $v_0(t)$. The condition indicated holds, if the control $v_0(t)$ assumes the value v_1 for $F_2(t) \geq 0$ and the value v_2 for $F_2(t) < 0$. It follows from (1.9) that the function $F_2(t)$ cannot assume only negative or only nonnegative values on the segment $[0, T]$. In accordance with (2.2) and (3.2) the function $F_2(t)$ increases in the interval $[0, t_0)$ and decreases in the interval $(t_0, T]$. In addition, the inequality $F_2(0) < F_2(T)$ holds. It follows from the enumerated properties of the function $F_2(T)$ that only two optimal accretion regimes are possible:

$$v_0(t) = v_2,\ 0 \leqslant t < t_1;\ v_0(t) = v_1,\ t_1 \leqslant t \leqslant T; \tag{3.3}$$

$$v_0(t) = v_2,\ 0 \leqslant t < t_2,\ t_3 \leqslant t \leqslant T;\ v_0(t) = v_1,\ t_2 \leqslant t < t_3, \tag{3.4}$$

where $t_1 < t_0$, $t_2 < t_0 < t_3$ are the moments at which the optimal rate of inflow of material changes.

We shall determine the conditions under which the accretion regime (3.3) is realized. The parameter t_1 is determined from (1.5) in the form

$$t_1 = (\lambda - v_1 T)(v_2 - v_1)^{-1}. \tag{3.5}$$

The inequality $F_2(T) \geq 0$ is the necessary and sufficient condition for realization of the accretion regime (3.3). According to (2.2), (2.3), (3.2), (3.3), and (3.5), we have

$$4\left[2(1+\lambda) + v_1\left(p_0 u_1^{-1} - T\right)\right]^{-1} \leqslant (1+\lambda)^{-1} + (v_2 - v_1)\left[(1+\lambda)v_2 - v_1(1 + v_2 T)\right]^{-1}. \tag{3.6}$$

If the inequality (3.6) does not hold, then the optimal accretion regime has the form (3.4). The parameters t_2 and t_3 are determined from the relations

$$\begin{aligned} t_3 - t_2 = (v_2 T - \lambda)(v_2 - v_1)^{-1},\ 2[1 + v_2 t_2 + v_1(t_0 - t_2)]^{-1} = \\ = (1 + v_2 t_2)^{-1} + [1 + v_2 t_2 + v_1(t_3 - t_2)]^{-1}. \end{aligned} \tag{3.7}$$

The first equality in (3.7) expresses the isoperimetric condition (1.5). The second relation in (3.7) follows from the equalities $F_2(t_2) = F_2(t_3) = 0$.

The relations obtained hold for $v_1 > 0$. If $v_1 = 0$, then they simplify substantially. In particular, the estimate (3.6) assumes the form

$$p_0 u_1^{-1} \geqslant \lambda v_2^{-1}, \tag{3.8}$$

while the parameters t_2 and t_3 are determined by the expressions $t_2 = (1/2)(p_0 u_1^{-1} + \lambda v_2^{-1})$, $t_3 = T + (1/2)(p_0 u_1^{-1} - \lambda v_2^{-1})$.

We shall formulate the results obtained in the form of a theorem.

THEOREM. If the inequality (3.6) holds, then the optimal rate of accretion of material is a piecewise-constant function with one switching point. The optimal rate of inflow of matter assumes the maximum value on the first interval of constancy and the minimum value on the second interval. If the inequality (3.6) does not hold, then the optimal rate of accretion is a piecewise-constant function with two switching points. The optimal rate of inflow of matter assumes the maximum value on the first and last intervals of constancy and the minimum value on the second interval.

4. Possible Generalizations. The theorem also permits determining the optimal rate of inflow of matter in a number of other problems of growth of elastic bodies. As an example we shall examine the problem of optimizing the twisting of a growing cylinder. Prior to the strain the cylinder with a circular transverse cross section with radius a_0 is in a natural state. At the time $t = 0$ a torque $M(t)$, $M(0) = 0$, is applied to its ends, and continuous accretion of matter starts on the unloaded lateral surface. The rate of change of the torque is bounded ($|M^{\cdot}(t)| \leq U_1$), and its value is given at the moment accretion stops [$M(T) = M_0$]. The problem is to find the admissible rate of inflow of matter $v_0(t)$ that ensures a fixed value of the radius of the cylinder at the moment accretion stops a_1 and gives a minimum for the angle of twist α at the final moment with an arbitrary admissible change of the torque, i.e., it gives $\inf_{v(t)} \sup_{u(t)} |\alpha(T)|$, where $u(t) = 2M^{\cdot}(t)(\pi G a_0^4)^{-1}$ and G is the constand shear modulus of the material.

For simplicity we restrict ourselves to the analysis of the process of accretion without any interference, when the minimum possible rate of inflow of matter equals zero. Repeating the arguments of Secs. 2 and 3 we find that the optimal rate of accretion is determined from the theorem, and the inequality characterizing the number of switching points has a form analogous to (3.8):

$$m_0 u_1^{-1} \geqslant \lambda v_2^{-1} \left(m_0 = 2M_0 \left(\pi G a_0^4\right)^{-1}\right).$$

I am deeply grateful to N. Kh. Arutyunyan for his interest in this work and for valuable remarks.

LITERATURE CITED

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